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Layered Von Kármán's swirling flow

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Abstract

Existence of an exact solution of the Navier–Stokes equations representing swirling flow consisting of two fluid layers is proven. It is assumed that the fluid in the bottom layer is injected into the swirl from a rotating disk. This describes the case when a disk melts in an ambient fluid and the melted fluid is removed by rotation of the disk.

Keywords: Navier–Stokes; Viscous flow; Rotating disk; Melting; Von Kármán swirling flow

1. Introduction

Von Kármán's swirling flow $U = (u, v, w)$ is a steady flow above a rotating disk. It can be described as an exact solution of the incompressible Navier–Stokes equations

$$U_t + (U \cdot \nabla)U = -\nabla p/\rho + \nu \Delta U,$$
$$\nabla \cdot U = 0$$

in the half-space $z > 0$ given by

$$u(x, y, z) = \Omega\big(f'(\zeta)x - g(\zeta)y\big),$$
$$v(x, y, z) = \Omega\big(f'(\zeta)y + g(\zeta)x\big),$$
$$w(x, y, z) = -2\sqrt{\Omega\nu}\, f(\zeta),$$
$$p(x, y, z) = -2\rho\Omega\nu\big(f'(\zeta) + f^2(\zeta)\big),$$

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where $\zeta = z\sqrt{\Omega/\nu} + \zeta_0$ and f, g satisfy

$$f''' + 2ff'' - f'^2 + g^2 = 0, \tag{1}$$

$$g'' + 2fg' - 2f'g = 0. \tag{2}$$

Ω is the angular velocity of the disk, p denotes the pressure, ρ is the density and ν the viscosity of the fluid. The above formulas for U and p are due to Von Kármán [10]. The first proof of existence of physically relevant solutions of (1), (2) is due to Von Kármán and Lin [11] and Howard [2] in case of a sufficiently large suction on the disk and more generally to McLeod [3]. A review of many related results that followed was done by Zandbergen and Dijkstra [13].

Here we are going to study the case when the swirl consists of two fluid layers. The bottom layer, $0 < z < d$, consists of a fluid coming out of the rotating disk and is assumed to have density $\bar{\rho}$ and viscosity $\bar{\nu}$. The top fluid has density ρ and viscosity ν. Let $\delta = d\sqrt{\Omega/\bar{\nu}}$ and choose ζ_0 so that ζ changes continuously at $z = d$, i.e. $d\sqrt{\Omega/\nu} + \zeta_0 = \delta$. On the interface $\zeta = \delta$, we require no cross flow ($w = 0$) and continuity of U, hence

$$f(\delta-) = f(\delta+) = 0, \qquad f'(\delta-) = f'(\delta+), \qquad g(\delta-) = g(\delta+). \tag{3}$$

Balancing the shear stress on the interface

$$\bar{\nu}\bar{\rho}u_z(x, y, d-) = \nu\rho u_z(x, y, d+), \qquad \bar{\nu}\bar{\rho}v_z(x, y, d-) = \nu\rho v_z(x, y, d+)$$

implies

$$f''(\delta-)/\lambda = f''(\delta+) \quad \text{and} \quad g'(\delta-)/\lambda = g'(\delta+), \tag{4}$$

where $\lambda = \sqrt{\nu/\bar{\nu}}\rho/\bar{\rho}$. At $z = 0$ the fluid is injected with velocity $W > 0$, i.e. $f(0) = -m$ where $m = W/(2\sqrt{\Omega\bar{\nu}})$. The no-slip condition at $z = 0$ implies $f'(0) = 0$ and $g(0) = 1$. By setting lateral velocities to zero at infinity we arrive at boundary conditions

$$f(0) = -m, \qquad f'(0) = 0, \qquad g(0) = 1, \qquad f'(\infty) = 0, \qquad g(\infty) = 0. \tag{5}$$

Therefore, we need to solve the boundary value problem (1), (2), (5) with a transition (3), (4) at some unknown δ.

Layered swirling flows can occur, for example, when the rotating disk melts or evaporates in an ambient fluid in chemical engineering processing, mass transfer, food processing, reactor cooling, and transpiration cooling. The melt may be removed by centrifugal forces, which has definite advantages over other means such as by blowing or by gravity. However, there are very few studies of effects of rotation on melting. Butuzov and Rifert [1] considered an approximate balance of viscous and centrifugal forces, while Wang [12] studied the rotation of a melted film using the above equations. In both cases the effect of the ambient fluid (air or liquid) is ignored ($\lambda = 0$). On the other hand, there exist studies on the spin coating process whereby a fluid film is depleted by rotation. It was found that the interfacial shear between the film and ambient air may be important [6,9]—melting or evaporation was not considered.

2. Main result

We are going to employ a shooting method to find suitable $f''(0)$ and $g'(0)$ that will solve the boundary value problem. Hence, we will study (1), (2) subject to initial conditions

$$\begin{aligned} &f(0) = -m, \qquad f'(0) = 0, \qquad f''(0) = \alpha, \\ &g(0) = 1 + \eta\beta, \qquad g'(0) = \beta \end{aligned} \tag{6}$$

where $m > 0$, $\eta \geqslant 0$ are arbitrary but fixed and α, β are real variables. η has no physical meaning in applications mentioned—it just makes the proof easier.

The initial value problem (1), (2), (6) has a solution on some maximal interval $[0, \ell_1)$ where $\ell_1 = \ell_1(\alpha, \beta) \in (0, \infty]$. If $f(\zeta) < 0$ for all $\zeta \in [0, \ell_1)$ define $\ell = \ell_1$. If on the other hand there exist a point $\delta \in (0, \ell_1)$ such that $f(\delta) = 0$ then we say that *transition happened* and at the first such point δ we redefine f, g for $\zeta \geqslant \delta$ to be a solution of (1), (2) on the maximal interval $[\delta, \ell)$ subject to initial conditions (3), (4) at δ. Let us denote by g'_l, g'_r to be left and right derivatives of g, hence, we require that $g'_l(\delta)/\lambda = g'_r(\delta)$.

The following Theorem shows that we can find α, β such that the transition happens, $\ell = \infty$ and boundary conditions (5) are satisfied.

Theorem 2.1. *For any* $\lambda, m \in (0, \infty)$, $\eta \geqslant 0$ *there exist* $\alpha > 0$, $\beta < 0$, $\delta > 0$ *such that*

(1) *the initial value problem* (1), (2), (6) *has a solution on* $[0, \delta]$ *and* $f(\delta) = 0$;
(2) *the initial value problem* (1)–(4) *has a solution on* $[\delta, \infty)$;
(3) $f' > 0$, $g > 0$, $g'_r < 0$ *on* $(0, \infty)$ *and for some* $k \in (0, \infty)$ *each of*

$$f'''(\zeta)e^{k\zeta}, \quad f''(\zeta)e^{k\zeta}, \quad f'(\zeta)e^{k\zeta}, \quad (k/2 - f(\zeta))e^{k\zeta}, \quad g'(\zeta)e^{k\zeta}, \quad g(\zeta)e^{k\zeta} \tag{7}$$

has a finite limit as $\zeta \to \infty$.

The proof of this theorem uses many different ideas and many of them can be found McLeod's paper [3]. A study of rough surfaces in [8] led to the introduction of η, which is used here just to simplify the proof. After we establish transition we can do asymptotics as in [8].

3. Proof of the theorem

Let us start with studying the solution f, g at a fixed $m > 0$, $\eta \geqslant 0$, $\alpha \in R$, $\beta \in R$. We shall frequently use the following obvious identities:

$$\left(f''e^F\right)' = \left(f'^2 - g^2\right)e^F, \tag{8}$$

$$\left(g'e^F\right)' = 2f'ge^F, \tag{9}$$

$$\left(f'''e^F\right)' = -2gg'e^F, \tag{10}$$

where $F(x) = 2\int_0^x f(t)\,dt$. Define $h_\delta = 1$ if transition does not happen. If transition happens at δ let $h_\delta(x) = 1$ for $x < \delta$ and $h_\delta(x) = \lambda$ for $x > \delta$. Note that integration of Eqs. (8) and (9) implies that, for $x \neq \delta$,

$$h_\delta(x) f''(x) e^{F(x)} - \alpha = \int_0^x h_\delta(s)\big(f'(s)^2 - g(s)^2\big) e^{F(s)}\, ds, \tag{11}$$

$$h_\delta(x) g'(x) e^{F(x)} - \beta = 2\int_0^x h_\delta(s) f'(s) g(s) e^{F(s)}\, ds. \tag{12}$$

Lemma 3.1. *If $f' \geqslant 0$, $g \geqslant 0$, $g_r' \leqslant 0$ on $[0, \ell)$ then*

(a) $\ell = \infty$,
(b) $f' > 0$, $g > 0$, $g_r' < 0$ *on* $(0, \infty)$,
(c) *transition happens,*
(d) $g(\infty) = 0$, $g'(\infty) = 0$, $0 \leqslant f''(\infty) < \infty$ *and if* $f''(\infty) = 0$ *then for some* $k \in (0, \infty)$ *each term in* (7) *has a finite limit as* $x \to \infty$.

Proof. Note first that there exists $\lim_{x\to\ell} g(x) \in [0, g(0)]$. (12) implies that

$$\beta \leqslant h_\delta g' e^F \leqslant 0. \tag{13}$$

This and (10) imply $|(f''' e^F)'| \leqslant d_1$, where $d_1 = 2g(0)|\beta| \max\{1, 1/\lambda\}$, hence

$$\big| f'''(x) e^{F(x)} - f'''(0)\big| \leqslant d_1 x \quad \text{for } x \in [0, \ell). \tag{14}$$

If $\ell < \infty$ then

$$\big| f'''(x) e^{-2xm}\big| \leqslant \big| f'''(0)\big| + d_1 x \tag{15}$$

and therefore f''' is bounded on $[0, \ell)$, which implies that f'', f', f have finite limits as $x \to \ell$. (13) implies that g' is bounded and hence g'' given by (2) is bounded, which implies that g' has a finite limit as $x \to \ell$ and therefore the solution of (1)–(6) can be continued which contradicts assumption $\ell < \infty$. Therefore $\ell = \infty$.

If $g(b) = 0$ for some $b > 0$ then $g \geqslant 0$ implies $g'(b) = 0$ and hence (2) implies contradiction $g(x) = 0$ for all $x \geqslant 0$. If $f'(b) = 0$ for some $b > 0$ then $f''(b) = 0$, but (1) implies $f'''(b) = -g(b)^2 < 0$ which is not possible since $f' \geqslant 0$. If $g_r'(b) = 0$ for some $b > 0$ then $g''(b) = 0$ and hence (2) and $f'(b) > 0$ imply contradiction $g(b) = 0$.

Suppose now that the transition does not happen, i.e. $f < 0$ on $[0, \infty)$. Note that (10) implies that $f''' e^F$ is nondecreasing.

<u>If $f'''(x_0) > 0$ for some $x_0 \geqslant 0$</u> then we should have $f'''(x) > f'''(x_0)$ for $x > x_0$ because F is decreasing and $f''' e^F$ is nondecreasing, which would imply cubic growth of f, contradicting $f < 0$.

<u>If $f'''(x_0) = 0$ for some $x_0 \geqslant 0$</u> then $f'''(x) = 0$ for $x \geqslant x_0$ because $f''' e^F$ is nondecreasing and f''' cannot be positive. $f'''(x) = 0$ for $x \geqslant x_0$ implies that for some constants a, b, c we have

$$f''(x) = a, \qquad f'(x) = ax + b, \qquad f(x) = ax^2/2 + bx + c \quad \text{for } x \geqslant x_0.$$

$f' \geqslant 0$ implies $a \geqslant 0$. This and $f < 0$ imply $a = 0$ and $b \leqslant 0$. Using $f' \geqslant 0$ again implies $b = 0$, $f = c$, $g = 0$ by (1) and hence $g = 0$ on $[0, \infty)$ by (2), which contradicts (6) and therefore the only possibility left is $\underline{f'''(x) < 0 \text{ for all } x \geqslant 0}$. $f''(x_0) \leqslant 0$ for some $x_0 \geqslant 0$ and $f''' < 0$ contradict $f' \geqslant 0$. On the other hand $f'' > 0$ and $f'(0) = 0$ contradict $f < 0$ and thus transition has to happen at some $\delta \in (0, \infty)$.

Since $g \geqslant 0$, $g_r' \leqslant 0$ we have that $g(\infty) \in [0, g(0)]$.

(2) implies that $g'' \geqslant 0$ on (δ, ∞) hence $g'(\infty) = 0$.

Pick $x_1 > \delta$. Since $f' > 0$ we have that $f(x_1) > 0$. Thus, $F(x) \geqslant 2(x - x_1)f(x_1) + F(x_1)$ for $x > x_1$ and (14) implies exponential decay of f'''. Therefore on $[\delta, \infty)$ we have

$$f''(x) = c_2 - \int_x^\infty f'''(t)\,dt,$$

$$f'(x) = c_1 + c_2 x - \int_x^\infty (x - t) f'''(t)\,dt, \tag{16}$$

$$f(x) = \frac{k}{2} + c_1 x + \frac{1}{2} c_2 x^2 - \frac{1}{2} \int_x^\infty (x - t)^2 f'''(t)\,dt,$$

$$F(x) = c_0 + kx + c_1 x^2 + \frac{1}{3} c_2 x^3 - \frac{1}{3} \int_x^\infty (x - t)^3 f'''(t)\,dt \tag{17}$$

for some finite constants c_i and k.

If $f(\infty) < \infty$ then $c_1 = c_2 = 0$ and (1) implies that $g(\infty) = 0$. (9) implies that $g'e^F$ is nondecreasing and since it is not positive it has a limit. This and (17) imply that $g'(x)e^{kx} \to a_0$ and hence $g(x)e^{kx} \to -a_0/k$. Thus the right hand side of (10) is integrable, hence $f'''e^F$ has a finite limit and therefore $f'''(x)e^{kx} \to a_1$ as $x \to \infty$. Which then implies $f''(x)e^{kx} \to -a_1/k$, $f'(x)e^{kx} \to a_1/k^2$ and $(k/2 - f(x))e^{kx} \to a_1/k^3$ as $x \to \infty$.

If $f(\infty) = \infty$ then either c_1 or c_2 is not 0 and hence $xf'(x)/f(x)$ converges to either 1 or 2 as $x \to \infty$. Thus there exist $\tau > 0$ and $a > \delta$ such that $xf'(x) > \tau f(x) > 0$ for $x > a$. If $g(\infty) > 0$ then (9) implies that for $x > a$

$$\begin{aligned} g'(x)e^{F(x)} - g'(a)e^{F(a)} &= 2\int_a^x f'ge^F > 2g(\infty)\tau \int_a^x \frac{f(u)}{u} e^{F(u)}\,du \\ &> \frac{g(\infty)\tau}{x}\left(e^{F(x)} - e^{F(a)}\right), \\ g'(x) > \frac{g(\infty)\tau}{x} + \left(g'(a) - \frac{g(\infty)\tau}{x}\right)&e^{F(a)-F(x)} \end{aligned}$$

which implies contradiction $g(\infty) = \infty$ and therefore we must have $g(\infty) = 0$. If $f''(\infty) = 0$ then $c_2 = 0$, $c_1 \neq 0$ and the exponential decay of f''' implies exponential decay of ff'', hence (1) implies $f'(\infty) = 0$—which contradicts $c_1 \neq 0$ and therefore $f''(\infty) = c_2 \neq 0$. $f' \geqslant 0$ and (16) imply $c_2 > 0$.

This completes the proof of the Lemma 3.1. □

Lemma 3.2. *The set of α, β, η for which $f' \geqslant 0$, $g \geqslant 0$, $g'_r \leqslant 0$ on $[0, \infty)$ is closed in R^3.*

Proof. Let f_n, g_n be solutions corresponding to $\alpha_n, \beta_n, \eta_n$ such that $f'_n \geqslant 0$, $g_n \geqslant 0$, $g'_{nr} \leqslant 0$ on $[0, \infty)$ and suppose that $\alpha_n, \beta_n, \eta_n$ converge to α, β, η. We want to show that the solution f, g corresponding to α, β, η also satisfies $f' \geqslant 0$, $g \geqslant 0$, $g'_r \leqslant 0$ on $[0, \infty)$.

Note that another integration of (11), (12) implies

$$f'_n(t) = \int_0^t \left(\alpha_n + \int_0^x h_{\delta_n}(s)\big(f'_n(s)^2 - g_n(s)^2\big)e^{F_n(s)}\, ds \right) h_{\delta_n}^{-1}(x)e^{-F_n(x)}\, dx, \tag{18}$$

$$g_n(t) - 1 - \eta_n\beta_n = \int_0^t \left(\beta_n + 2\int_0^x h_{\delta_n}(s) f'_n(s) g_n(s) e^{F_n(s)}\, ds \right) h_{\delta_n}^{-1}(x)e^{-F_n(x)}\, dx. \tag{19}$$

(13) implies that there exists $c_1 < \infty$ such that for all n

$$\left|g'_n(x)\right| \leqslant c_1 e^{2mx} \quad \text{for } x \neq \delta_n \tag{20}$$

hence the Arzela–Ascoli Theorem (a version of [7, p. 5] will do) implies that there exists $\tilde{g} \in C[0, \infty)$ such that a subsequence of g_n converges uniformly to $\tilde{g}$ on every finite interval. Let us restart with this subsequence.

(15) implies that there exists $c_2 < \infty$ such that for all n

$$\left|f'''_n(x)\right| \leqslant c_2 e^{3mx} \quad \text{for } x \neq \delta_n \tag{21}$$

which implies the same kind of estimates for f''_n and f'_n. Hence we can apply again Arzela–Ascoli Theorem to conclude that there exists $\phi \in C[0, \infty)$ such that $f'_{n_i} \to \phi$ uniformly on $[0, L]$ for all $L > 0$. Define $\tilde{f}(x) = -m + \int_0^x \phi$. Hence

$$f_n \to \tilde{f}, \quad f'_n \to \tilde{f}', \quad g_n \to \tilde{g} \quad \text{uniformly on } [0, L] \text{ for all } L > 0 \tag{22}$$

—after replacing the original sequence with a subsequence. Clearly,

$$\tilde{f}' \geqslant 0, \qquad \tilde{g} \geqslant 0. \tag{23}$$

It will be now shown that δ_n form a bounded sequence. If this would not be the case, then we could assume that $\delta_n \to \infty$—after replacing the original sequence with a subsequence. Taking a limit in (18), (19) would give integral equations for $\tilde{f}, \tilde{g}$; differentiation would imply that $\tilde{f}, \tilde{g}$ satisfy (1), (2), (6) on $[0, \infty)$. Taking the derivative of (19) would give that $g'_n \to \tilde{g}'$, hence $\tilde{g}' \leqslant 0$; (23) and Lemma 3.1 would then imply that $\tilde{f}(\tilde{\delta}) = 0$ for some $\tilde{\delta} \in (0, \infty)$. $\delta_n \to \infty$ implies $\tilde{f} \leqslant 0$ but since $\tilde{f}' \geqslant 0$ this would imply $\tilde{f}(x) = 0$ for $x > \tilde{\delta}$. This and (1), (2) would then imply contradiction $\tilde{g} \equiv 0$.

Now we will show that δ_n are bounded away from 0. Since $f(0) < 0$, we have that $f < 0$ on $[0, b]$ for some $b > 0$. Continuous dependence on initial conditions implies that $\delta_n > b$ for all large enough n.

Therefore a subsequence of δ_n converges to some $\tilde{\delta} \in (0, \infty)$ and hence restarting the argument with this subsequence we may assume that $\delta_n \to \tilde{\delta}$. Being a limit we have to have that $\tilde{f} \leqslant 0$ on $[0, \tilde{\delta})$ and $\tilde{f} \geqslant 0$ on $(\tilde{\delta}, \infty)$ and in particular $\tilde{f}(\tilde{\delta}) = 0$. Note that $h_{\delta_n} \to h_{\tilde{\delta}}$

and hence we can take the limit as $n \to \infty$ in (18), (19) giving us integral equations for $\tilde{f}'$, $\tilde{g}$ which then imply that $\tilde{f}$ and $\tilde{g}$ satisfy (1), (2), (6) on $[0, \tilde{\delta}]$ as well as (1)–(4) on $[\tilde{\delta}, \infty)$. Since $\tilde{f}(0) < 0$, $\tilde{f}(\tilde{\delta}) = 0$, $\tilde{f} \leqslant 0$ and $\tilde{f}' \geqslant 0$ on $[0, \tilde{\delta}]$ we have that $\tilde{f} < 0$ on $[0, \tilde{\delta})$. Therefore $\tilde{f} = f$, $\tilde{g} = g$ and $\tilde{\delta} = \delta$. Differentiation of (19) and letting $n \to \infty$ implies $g_n' \to g'$ and hence $g' \leqslant 0$ away from δ.

This completes the proof of Lemma 3.2. $\square$

Fix $\eta \geqslant 0$.

Define S to be the set of pairs (α, β) such that $f' \geqslant 0$, $g \geqslant 0$, $g_r' \leqslant 0$ on $[0, \infty)$.

Define S_0 to be the set of pairs $(\alpha, \beta) \in S$ such that $f''(\infty) = 0$. In view of Lemma 3.1, it is enough to prove that S_0 is not empty.

Define S^+ to be the set of pairs (α, β), with $\alpha > 0$, such that there exists $x^+ \in (0, \ell)$ for which $g_r'(x^+) > 0$, $g > 0$ on $[0, x^+]$ and $f' > 0$ on $(0, x^+]$.

Define S^- to be the set of pairs (α, β), with $\alpha > 0$, such that there exists $x^- \in (0, \ell)$ for which $g(x^-) < 0$, $g_r' < 0$ on $[0, x^-]$ and $f' > 0$ on $(0, x^-]$.

Clearly S, S^+, S^- are mutually disjoint and S is closed by Lemma 3.2.

Lemma 3.3. *S^- and S^+ are open sets in the plane.*

Proof. Suppose $(\alpha, \beta) \in S^+$ and let x^+ be as in the definition. If transition does not happen or if transition happens at some δ but $x^+ \leqslant \delta$ then continuous dependence of the solutions of ODEs on initial values and $\alpha > 0$ imply persistence of x^+ in a neighborhood of (α, β).

If transition happens at some δ but $x^+ > \delta$ then $f'(\delta) > 0$ implies that δ is a continuous function of α, β hence we can choose $x^+ > \delta$ in a neighborhood of (α, β) just as above.

This shows openness of S^+. Exactly the same argument works for S^-. $\square$

Lemma 3.4. *If $\alpha > 0$ and $\beta = 0$ then $(\alpha, \beta) \in S^+$.*

Proof. Since $f''(0) = \alpha > 0$, $f(0) = -m < 0$ and $g(0) = 1$ there exists $x^+ > 0$ such that $f'' > 0$, $f < 0$ and $g > 0$ on $[0, x^+]$. Hence $f' > 0$ on $(0, x^+]$ and (9) implies $(g'e^F)' > 0$ on $(0, x^+]$ and therefore $g' > 0$ on $(0, x^+]$. $\square$

Lemma 3.5. *If $\eta > 0$, $\alpha > 0$ and $\beta = -1/\eta$ then $(\alpha, \beta) \in S^-$.*

Proof. Since $g'(0) = \beta < 0$, $f''(0) = \alpha > 0$ and $f(0) = -m < 0$ there exists $x^- > 0$ such that $g' < 0$, $f'' > 0$ and $f < 0$ on $[0, x^-]$; and since $g(0) = f'(0) = 0$ we have that $g < 0$ and $f' > 0$ on $(0, x^-]$. $\square$

Lemma 3.6. *$(0, \beta) \notin S$ whenever $\beta \in (-\infty, \infty)$.*

Proof. If $\alpha = 0$, $\beta \neq -1/\eta$ then $f'(0) = f''(0) = 0$, $f'''(0) = -g(0)^2 < 0$; hence $f'(\tau) < 0$ for some $\tau > 0$. When $\alpha = 0$ and $\beta = -1/\eta$ then $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = 0$ and $f^{(5)} = -2\beta^2$, hence $f'(\tau) < 0$ for some $\tau > 0$. $\square$

Lemma 3.7. *If $\alpha > 0$, $\beta \in (-1/\eta, 0)$, $(\alpha, \beta) \notin S^+ \cup S^-$ and $f' > 0$ on $(0, a)$ for some $a \in (0, \ell)$, then, $g > 0$ and $g_r' < 0$ on $[0, a)$. Moreover, if in addition $f'(a) = 0$ then $f' < 1$ on $[0, a]$.*

Proof. If $g(b) \leqslant 0$ for some $b \in (0, a)$ pick the smallest such b. Hence $g(b) = 0$, $g_l'(b) \leqslant 0$ and since $g \not\equiv 0$ we have that $g_l'(b) < 0$. (9) implies that $g_l' < 0$ on $[0, b]$ implying contradiction $(\alpha, \beta) \in S^-$. Therefore $g > 0$ on $[0, a)$.

If $g_r'(b) \geqslant 0$ for some $b \in (0, a)$ then (9) implies that $g_r' > 0$ on (b, a) and hence $(\alpha, \beta) \in S^+$ which is a contradiction. Therefore $g_r' < 0$ on $[0, a)$.

Suppose now $f'(a) = 0$. Since $f''(0) > 0$ there exists $b \in (0, a)$ where f' attains a local maximum. Hence $f''(b) = 0$, $f'''(b) \leqslant 0$ and (1) imply that $f'(b)^2 \leqslant g(b)^2 < g(0)^2$ and therefore $f' < g(0) < 1$ on $[0, a]$. □

Lemma 3.8. *If $\alpha \geqslant 6 \max\{1, \lambda\}$, $\beta \in (-1/\eta, 0)$ and $(\alpha, \beta) \notin S^+ \cup S^-$ then $(\alpha, \beta) \in S$.*

Proof. Suppose $(\alpha, \beta) \notin S$. Lemma 3.7 implies that there exists $a \in (0, \ell)$ such that $f' > 0$ on $(0, a)$, $f'(a) = 0$, $f' < 1$ on $[0, a]$, hence, $f''(a-) \leqslant 0$, $F(x) \leqslant x^2$ and (11) implies

$$\left|h_\delta(x) f''(x) e^{F(x)} - \alpha\right| \leqslant \int_0^x h_\delta(u) e^{u^2}\, du \quad \text{for } x \in [0, a],\ x \neq \delta. \tag{24}$$

If $a \leqslant 1$ then (24) implies

$$h_\delta(a) f''(a-) e^{F(a)} \geqslant \alpha - e \max\{1, \lambda\} > 0$$

implying contradiction $f''(a-) > 0$.

If $a > 1$ then (24) implies, except at δ,

$$e \max\{1, \lambda\} f'' \geqslant h_\delta f'' e^F \geqslant \alpha - e \max\{1, \lambda\} \qquad \text{on } [0, 1],$$

$$f'' \geqslant (6 - e)/e > 1 \quad \text{on } [0, 1]$$

implying contradiction $f'(1) > 1$. Therefore there is no such a. □

Lemma 3.9. *If $\beta \in (-1/\eta, 0]$ and $(\alpha, \beta) \in S_0$ then $|\beta| \leqslant 1/(4m)$.*

Proof. Define $\phi = (f'')^2 + (g')^2$. Note that [4]

$$\phi'' + 2f\phi' = 2(f''')^2 + 2(g'')^2$$

hence $(\phi' e^F)' \geqslant 0$. Thus, if $\phi'(a) > 0$ then $\phi'(x) > 0$ for all $x \geqslant a$. So, when $(\alpha, \beta) \in S_0$ then $\phi \to 0$, hence $\phi'(x) \leqslant 0$ for all $x \neq \delta$, and in particular,

$$\phi'(0) = 4m\alpha^2 - 2\alpha(1 + \eta\beta)^2 + 4m\beta^2 \leqslant 0,$$

$$\left(4m\alpha - (1 + \eta\beta)^2\right)^2 + (4m\beta)^2 - (1 + \eta\beta)^4 \leqslant 0,$$

$$(4m\beta)^2 \leqslant (1 + \eta\beta)^4 \leqslant 1. \qquad \square$$

Lemma 3.10. *$S \setminus S_0$ is relatively open in the complement of $S^+ \cup S^-$ in $\alpha > 0$.*

Proof. Suppose $(\alpha_0, \beta_0) \in S \setminus S_0$, $\alpha_0 > 0$ and let f_0, g_0 be the corresponding solution. Then $f_0'(a) > 1$ and $f_0''(a) > 0$ for some $a > 0$ hence continuity implies that for all (α, β) close enough to (α_0, β_0) we can assume that $f' > 0$ on $(0, a)$, $f'(a) > 1$ and $f''(a) > 0$. Hence if $(\alpha, \beta) \notin S^+ \cup S^-$ then Lemma 3.7 implies that $(\alpha, \beta) \in S$. (8) then implies that $f'' > 0$ and hence $f' > 1$ on $[a, \infty)$ hence $f'(\infty) \neq 0$ and therefore $f''(\infty) > 0$. $\square$

Proof of Theorem 2.1. Assume first that $\eta > 0$.

Lemmas 3.6, 3.2 imply that there exists $\alpha_l > 0$ such that $(\alpha, \beta) \notin S$ whenever $\alpha \in [0, \alpha_l]$ and $\beta \in [-1/\eta, 0]$. Consider a rectangle $\alpha_l \leqslant \alpha \leqslant \alpha_r \equiv 6 \max\{1, \lambda\}$, $-1/\eta \leqslant \beta \leqslant 0$. The top $\beta = 0$ belongs to an open set S^+ by Lemma 3.4. The bottom $\beta = -1/\eta$ belongs to an open set S^- by Lemma 3.5. A classical separation theorem, see [5], asserts that there exists a continuum (i.e. closed connected set) in the rectangle that contains a point on the line $\alpha = \alpha_l$ as well a point on the line $\alpha = \alpha_r$ and lies in the complement of $S^+ \cup S^-$.

Since the line $\alpha = \alpha_l$ is in S^c, the complement of S, a part of the continuum is in S^c, which is open by Lemma 3.2. Another part lies in S_0 or $S \setminus S_0$ by Lemma 3.8. Since $S \setminus S_0$ is open relative to the continuum by Lemma 3.10 and the continuum is connected we have to have that S_0 cannot be empty. This proves the theorem when $\eta > 0$.

Pick $\eta_n > 0$ such that $\eta_n \to 0$ and then let (α_n, β_n) be the corresponding solutions in $S_0(\eta = \eta_n)$. Lemma 3.9 implies that we can assume, after renaming a subsequence, that $\alpha_n \to \alpha \in [0, \alpha_r]$, $\beta_n \to \beta \in [-1/(4m), 0]$. Lemma 3.2 implies that $(\alpha, \beta) \in S(\eta = 0)$. Lemma 3.6 implies $\alpha > 0$. To complete the proof it is enough to show $(\alpha, \beta) \in S_0(\eta = 0)$.

If $(\alpha, \beta) \in S \setminus S_0(\eta = 0)$ then $f'(a) > 1$ and $f''(a) > 0$ for some $a > 0$ hence continuity implies that for all large enough n we have that $f_n' > 0$ on $(0, a)$, $f_n'(a) > 1$ and $f_n''(a) > 0$. (8) then implies that $f_n'' > 0$ and hence $f_n' > 1$ on $[a, \infty)$ which contradicts $f_n'(\infty) = 0$. $\square$

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